Approximating the Early Exercise Boundary for American-style Options (Financial Modeling and Analysis)

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Approximating the Early Exercise Boundary for American-style Options*

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1 Introduction

European-style options, which can only be exercised at its maturity, have closed-form formulas for their values in the standard model pioneered by Black and Scholes [7] and Merton [25]. Although a vast majority of traded options are of American-style optimally exercised before maturity, there are no closed-form formulas for their values even in the standard model called *vanilla*. The original statements of the American options problem are dating back to the work of Samuelson [28] and McKean [24]; see Barone-Adesi [2] for a concise review of the American options problem. The principal difficulty in analyzing American options may be the absence of an explicit expression for the *early exercise boundary* (EEB), which is an optimal level of critical asset value where early exercise occurs; see Zhu [29, Equation (27)] for a complicated expression in an infinite-series form.

Kim [19], Jacka [16] and Carr, Jarrow and Myneni [11] provided the put value in integral form as a function of the EEB. To implement their approach, we need to obtain an accurate EEB approximation possibly in closed form. Various approximations have been developed by many researchers; see, e.g., Barone-Adesi and Whaley [3], Bunch and Johnson [8], Carr [9], Geske and Johnson [13], MacMillan [22], Zhu [30] and Zhu and He [31]. Among them, however, there is no explicit approximation for the EEB. No doubt, the simplest approximation is a flat boundary. Barone-Adesi and Whaley [3] proposed a flat approximation as an initial guess of their iterative procedure to find the optimal EEB. With the aid of this approximation, Bjerksund and Stensland [6] analyzed American options as barrier options with knockout feature. Huang, Subrahmanyam and Yu [15] assumed the EEB as a piecewise-constant function of time, and provided a recursive algorithm for obtaining the optimal exercise levels; see also Bjerksund and Stensland [6]. Alternatively, Omberg [27] developed an exponential EEB, and Ju [17] approximated the EEB as a piecewise-exponential function of time to maturity. In both approximations, however, there are no closed-form solutions for the bases and the exponents of those exponential functions, which must be computed numerically in their approaches. The multipiece EEB approximations in [15, 17] naturally have discontinuous points in the boundary, but the EEB should be smooth intrinsically [26]. Clearly, the discontinuity

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in the multipiece EEB approximations become an serious obstacle for accurate decision making of the option holders. The purpose of this paper is to approximate the EEB by a single exponential function with an explicit and asymptotically exact exponent.

This paper is organized as follows: To avoid prolixity, we primarily focus on the problem of valuing the American put option, but we provide the corresponding results for the associated American call case as well. In Section 2, we formulate the problem by a free boundary problem in the classical Black-Scholes-Merton framework to obtain a basic partial differential equation for the American put value. In Section 3, following Kimura [20], we adopt the Laplace-Carson transform approach to derive a functional equation for the transformed EEB, from which we obtain two different exponential EEB approximations in Section 4. In order to improve the accuracy of these approximations near expiry, we develop a heuristic refinement in Section 5.

2 Black-Scholes-Merton Formulation

Assume that the capital market is well-defined and follows the efficient market hypothesis. Let $(S_t)_{t\geq 0}$ be the asset price governed by the risk-neutralized diffusion process

$$\frac{\mathrm{d}S_t}{S_t} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geq 0, \tag{2.1}$$

where $r > 0$ is the risk-free interest rate, $\delta \geq 0$ is a continuous dividend rate, $\sigma > 0$ is a volatility of the asset returns. In (2.1), $(W_t)_{t\geq 0}$ is a standard Wiener process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to W and the probability measure $\mathbb{P}$ is chosen risk-neutrally so that the asset has mean rate of return r. We consider an American *put* option written on the asset price process $(S_t)_{t\geq 0}$, which has maturity $T > 0$ and strike price $K > 0$. Let

$$P \equiv P(t, S_t) = P(t, S_t; K, r, \delta), \quad 0 \leq t \leq T, \tag{2.2}$$

denote the value of the American put option at time t. Similarly, let $C \equiv C(t, S_t) = C(t, S_t; K, r, \delta)$ $(0 \leq t \leq T)$ denote the value of the associated American call option with the same parameters as those in the put option.

From the theory of arbitrage pricing, the fair value of the American put option at time t is given by solving an *optimal stopping problem*

$$P(t, S_t) = \operatorname*{ess\,sup}_{\mathcal{T}_t \in [t,T]} \mathbb{E}\left[\mathrm{e}^{-r(\mathcal{T}_t - t)}(K - S_{\mathcal{T}_t})^+ \,\middle|\, \mathcal{F}_t\right], \quad 0 \leq t \leq T, \tag{2.3}$$

where $\mathcal{T}_t$ is a stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. The random variable $\mathcal{T}_t^* \in [t, T]$ is called an *optimal stopping time* if it gives the supremum value of the right-hand side of

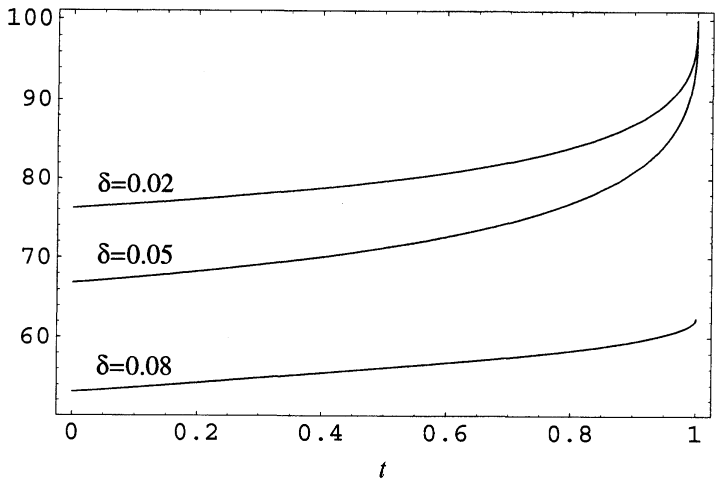

Figure 1: Early exercise boundaries $B_p(t)$ ($t \in [0,T]$) for American put options ($T=1$, $K=100$, $r=0.05$, $\delta=0.02, 0.05, 0.08$, $\sigma=0.2$)

(2.3). The relationship between the early exercise feature of American options and optimal stopping problems was first analyzed by McKean [24] who studied the problem (2.3) under an actual probability measure rather than $\mathbb{P}$. Mathematically rigorous treatment of the problem (2.3) was first established by Bensoussan [4] and Karatzas [18].

Solving the optimal stopping problem (2.3) is equivalent to find the points (t, S_t) for which early exercise is optimal. Let $\mathcal{S}$ and $\mathcal{C}$ denote the *stopping region* and *continuation region*, respectively. The stopping region $\mathcal{S}$ is defined by

$$\mathcal{S} = \left\{(t,S) \in [0,T] \times \mathbb{R}_+ \,|\, P(t,S) = (K-S)^+\right\}. \tag{2.4}$$

Of course, the continuation region $\mathcal{C}$ is the complement of $\mathcal{S}$ in $[0,T] \times \mathbb{R}_+$. The boundary that separates $\mathcal{S}$ from $\mathcal{C}$ is referred to as the *early exercise boundary* (EEB), which is defined by

$$B_p(t) = \sup\left\{S \in \mathbb{R}_+ \,|\, P(t,S) = (K-S)^+\right\}, \quad 0 \le t \le T. \tag{2.5}$$

Similarly, define the EEB for the American call option by

$$B_c(t) = \inf\left\{S \in \mathbb{R}_+ \,|\, C(t,S) = (S-K)^+\right\}, \quad 0 \le t \le T. \tag{2.6}$$

Between these two boundaries $B_p(t) \equiv B_p(t; r, \delta)$ and $B_c(t) \equiv B_c(t; r, \delta)$, Carr and Chesney [10] derived a simple symmetric relation such that

$$B_c(t; r, \delta) B_p(t; \delta, r) = K^2, \quad 0 \le t \le T. \tag{2.7}$$

McKean [24] showed that the American put value and the early exercise boundary can be obtained by jointly solving a *free boundary problem*, which is specified by the Black-Scholes-Merton partial differential equation (PDE)

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-\delta) S \frac{\partial P}{\partial S} - rP = 0, \quad S > B_p(t), \tag{2.8}$$

together with the boundary conditions

$$\left| \begin{array}{l} \lim_{S\uparrow\infty} P(t,S) = 0 \\ \lim_{S\downarrow B_p(t)} P(t,S) = K - B_p(t) \\ \lim_{S\downarrow B_p(t)} \dfrac{\partial P}{\partial S} = -1, \end{array} \right. \tag{2.9}$$

and the *terminal condition*

$$P(T,S) = (K-S)^+. \tag{2.10}$$

The second condition in (2.9) is often called the *value-matching condition*, while the third condition is called the *smooth-pasting* or *high-contact condition.*

It is sometimes convenient to work with the equations where the current time t is replaced by the time to expiry $\tau \equiv T - t$. For the sake of notational convenience, we write $\widetilde{S}_\tau \equiv S_{T-\tau} = S_t$ and $\widetilde{B}_p(\tau) \equiv B_p(T-\tau) = B_p(t)$, and we refer to $(\widetilde{S}_\tau)_{\tau\leq T}$ as the *backward running process* of $(S_t)_{t\geq 0}$. From (2.8)–(2.10), the put price for the backward running process $\widetilde{P}(\tau, \widetilde{S}_\tau) \equiv P(T-\tau, S_{T-\tau}) = P(t, S_t)$ satisfies the PDE

$$-\frac{\partial \widetilde{P}}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widetilde{P}}{\partial S^2} + (r-\delta)S\frac{\partial \widetilde{P}}{\partial S} - r\widetilde{P} = 0, \quad S > \widetilde{B}_p(\tau), \tag{2.11}$$

with the boundary conditions

$$\left| \begin{array}{l} \lim_{S\uparrow\infty} \widetilde{P}(\tau,S) = 0 \\ \lim_{S\downarrow \widetilde{B}_p(\tau)} \widetilde{P}(\tau,S) = K - \widetilde{B}_p(\tau) \\ \lim_{S\downarrow \widetilde{B}_p(\tau)} \dfrac{\partial \widetilde{P}}{\partial S} = -1, \end{array} \right. \tag{2.12}$$

and the *initial condition*

$$\widetilde{P}(0,S) = (K-S)^+. \tag{2.13}$$

Similarly, we can show that the call price for the backward running process $\widetilde{C}(\tau, \widetilde{S}_\tau) \equiv C(T-\tau, S_{T-\tau}) = C(t, S_t)$ satisfies the PDE

$$-\frac{\partial \widetilde{C}}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widetilde{C}}{\partial S^2} + (r-\delta)S\frac{\partial \widetilde{C}}{\partial S} - r\widetilde{C} = 0, \quad S < \widetilde{B}_c(\tau), \tag{2.14}$$

with the boundary conditions

$$\left| \begin{array}{l} \lim_{S\downarrow 0} \widetilde{C}(\tau,S) = 0 \\ \lim_{S\uparrow \widetilde{B}_c(\tau)} \widetilde{C}(\tau,S) = \widetilde{B}_c(\tau) - K \\ \lim_{S\uparrow \widetilde{B}_c(\tau)} \dfrac{\partial \widetilde{C}}{\partial S} = 1, \end{array} \right. \tag{2.15}$$

and the initial condition

$$\widetilde{C}(0,S) = (S-K)^+, \tag{2.16}$$

where $\widetilde{B}_c(\tau) \equiv B_c(T-\tau) = B_c(t)$.

3 Valuation in the Laplace Domain

3.1 Laplace-Carson Transforms

For $\lambda > 0$, define the Laplace-Carson transform (LCT) of the time-reversed quantities as

$$P^*(\lambda, S) = \mathcal{LC}[\widetilde{P}(\tau, S)] = \int_0^\infty \lambda \mathrm{e}^{-\lambda\tau} \widetilde{P}(\tau, S) \mathrm{d}\tau, \tag{3.1}$$

and $C^*(\lambda, S) = \mathcal{LC}[\widetilde{C}(\tau, S)]$. No doubt, there is no essential difference between the LCT and the Laplace transform (LT) defined by

$$\widehat{P}(\lambda, S) = \int_0^\infty \mathrm{e}^{-\lambda\tau} \widetilde{P}(\tau, S) \mathrm{d}\tau.$$

Clearly, we have $P^*(\lambda, S) = \lambda \widehat{P}(\lambda, S)$ for $\lambda > 0$. The principal reason why we prefer LCT to LT is that LCT generates relatively simpler formulas than LT for option pricing problems because constant values are invariant after taking transformation. In the context of option pricing, LCTs were first used in the *randomization* of Carr [9] for valuing an American vanilla put option with an exponentially distributed random maturity T. The idea of randomization gives us another interpretation that the LCT $P^*(\lambda, S)$ can be regarded as an exponentially weighted sum (integral) of the time-reversed value $\widetilde{P}(\tau, S)$ for (infinitely many) different values of the maturity $T \in \mathbb{R}_+$, and hence for $\tau \in \mathbb{R}_+$, which makes LCTs well defined.

3.2 European Options

For American vanilla options, it is well known that the value of an American option can be represented as the sum of the value of the corresponding European option and the *early exercise premium*. Kim [19] proved that the option value has such a decomposition and that the premium has an integral representation; see Kim [19, Equations (6) and (12)]. Here, as a preliminary for valuing American options, we derive closed-form LCTs of the European values.

Consider a vanilla European put option written on the asset price process $(S_t)_{t\geq 0}$ that has constant maturity T and strike price K. Let $p \equiv p(t, S_t)$ denote the value of the European put option at time $t \in [0, T]$. Then, in much the same way as in Section 2, the put value for the backward running process $\widetilde{p}(\tau, \widetilde{S}_\tau) \equiv p(T-\tau, S_{T-\tau}) = p(t, S_t)$ for $\tau = T - t$ can be obtained by solving the PDE

$$-\frac{\partial \widetilde{p}}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widetilde{p}}{\partial S^2} + (r-\delta) S \frac{\partial \widetilde{p}}{\partial S} - r\widetilde{p} = 0, \quad S > 0, \tag{3.2}$$

with the boundary conditions

$$\left| \begin{array}{l} \lim_{S \downarrow 0} \widetilde{p}(\tau, S) = K\mathrm{e}^{-r\tau} \\ \lim_{S \uparrow \infty} \widetilde{p}(\tau, S) = 0, \end{array} \right. \tag{3.3}$$

and the same initial condition as in (2.13), *i.e.*,

$$\widetilde{p}(0, S) = (K - S)^+. \tag{3.4}$$

For convenience, denote $p^*(\lambda, S) = \mathcal{LC}[\widetilde{p}(\tau, S)]$. We see from (3.2)–(3.4) that $p^*(\lambda, S)$ satisfies the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 p^*}{\mathrm{d}S^2} + (r-\delta) S \frac{\mathrm{d}p^*}{\mathrm{d}S} - (\lambda + r) p^* + \lambda (K - S)^+ = 0, \quad S > 0, \tag{3.5}$$

with the boundary conditions

$$\left| \begin{array}{l} \lim_{S \downarrow 0} p^*(\lambda, S) = \dfrac{\lambda K}{\lambda + r} \\ \lim_{S \uparrow \infty} p^*(\lambda, S) = 0. \end{array} \right. \tag{3.6}$$

Proposition 1 (Kimura [20]) *The LCT $p^*(\lambda, S)$ for the European put value is given by*

$$p^*(\lambda, S) = \begin{cases} \xi(S) + \dfrac{\lambda K}{\lambda + r} - \dfrac{\lambda S}{\lambda + \delta}, & S < K \\ \eta(S), & S \geq K, \end{cases}$$

where

$$\xi(S) = \frac{K}{\theta_1 - \theta_2} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r} \theta_2\right) \left(\frac{S}{K}\right)^{\theta_1}, \quad S < K$$

$$\eta(S) = \frac{K}{\theta_1 - \theta_2} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r} \theta_1\right) \left(\frac{S}{K}\right)^{\theta_2}, \quad S \geq K,$$

and the parameters $\theta_i \equiv \theta_i(\lambda)$ $(i = 1, 2,\ \theta_1 > 1,\ \theta_2 < 0)$ are two roots of the quadratic equation

$$\frac{1}{2}\sigma^2 \theta^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta - (\lambda + r) = 0, \tag{3.7}$$

i.e.,

$$\theta_i = \frac{1}{\sigma^2} \left\{ -(r - \delta - \tfrac{1}{2}\sigma^2) - (-1)^i \sqrt{(r - \delta - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)} \right\}, \quad i = 1, 2.$$

We can apply the same argument to the call case: Let $c \equiv c(t, S_t)$ be the value of the European call option at time $t \in [0, T]$, $\widetilde{c}(\tau, \widetilde{S}_\tau) \equiv c(T-\tau, S_{T-\tau}) = c(t, S_t)$ for $\tau = T - t$, and $c^*(\lambda, S) = \mathcal{LC}[\widetilde{c}(\tau, S)]$. Solving the ODE

$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 c^*}{\mathrm{d}S^2} + (r-\delta)S\frac{\mathrm{d}c^*}{\mathrm{d}S} - (\lambda + r)c^* + \lambda(S-K)^+ = 0, \quad S > 0, \tag{3.8}$$

together with the boundary conditions

$$\left| \begin{array}{l} \lim_{S \downarrow 0} c^*(\lambda, S) = 0 \\ \\ \lim_{S \uparrow \infty} \dfrac{\mathrm{d}c^*}{\mathrm{d}S} < \infty, \end{array} \right. \tag{3.9}$$

we have

Proposition 2 (Kimura [20]) *The LCT $c^*(\lambda, S)$ for the European call value is given by*

$$c^*(\lambda, S) = \begin{cases} \xi(S), & S < K \\ \eta(S) + \dfrac{\lambda S}{\lambda + \delta} - \dfrac{\lambda K}{\lambda + r}, & S \geq K. \end{cases}$$

3.3 American Options

Now we apply the argument above to the American put option. From (2.11)–(2.13), the LCT $P^*(\lambda, S)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 P^*}{\mathrm{d}S^2} + (r-\delta)S\frac{\mathrm{d}P^*}{\mathrm{d}S} - (\lambda + r)P^* + \lambda(K-S)^+ = 0, \quad S > B_p^*, \tag{3.10}$$

together with the boundary conditions

$$\left| \begin{array}{l} \lim_{S \uparrow \infty} P^*(\lambda, S) = 0 \\ \\ \lim_{S \downarrow B_p^*} P^*(\lambda, S) = K - B_p^* \\ \\ \lim_{S \downarrow B_p^*} \dfrac{\mathrm{d}P^*}{\mathrm{d}S} = -1, \end{array} \right. \tag{3.11}$$

where $B_p^* \equiv B_p^*(\lambda) = \mathcal{LC}[\widetilde{B}_p(\tau)]$, which is a constant in the Laplace world due to the memoryless property of the exponential distribution.

Theorem 1 *The LCT $P^*(\lambda, S)$ for the American put value is given by*

$$P^*(\lambda, S) = \begin{cases} K - S, & S \leq B_p^* \\ p^*(\lambda, S) + e_p^*(\lambda, S), & S > B_p^*, \end{cases}$$

where

$$e_p^*(\lambda, S) = -\frac{1}{\theta_2}\left\{\theta_1 \xi(B_p^*) + \frac{\delta}{\lambda + \delta} B_p^*\right\}\left(\frac{S}{B_p^*}\right)^{\theta_2}, \quad S > B_p^*,$$

and B_p^* $(\leq K)$ *is a unique positive solution of the functional equation*

$$\lambda\left(\frac{B_p^*}{K}\right)^{\theta_1}+\delta\theta_1\frac{B_p^*}{K}+r(1-\theta_1)=0. \tag{3.12}$$

Kim [19, Section 3] proved that the vanilla American put value has the decomposition

$$P(t,S_t)=p(t,S_t)+e_p(t,S_t), \tag{3.13}$$

and that the premium $e_p(t,S_t)$ has the integral representation

$$\begin{aligned}e_p(t,S_t)=\int_t^T &\{rK\mathrm{e}^{-r(u-t)}\Phi(-d_-(S_t,B_p(u),u-t))\\ &-\delta S_t\mathrm{e}^{-\delta(u-t)}\Phi(-d_+(S_t,B_p(u),u-t))\}\,\mathrm{d}u,\end{aligned} \tag{3.14}$$

where $\Phi(\,\cdot\,)$ is the cumulative standard normal distribution function, and for $x,y,\tau>0$

$$d_\pm(x,y,\tau)=\frac{\log(x/y)+(r-\delta\pm\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{3.15}$$

See also Jacka [16] and Carr et al. [11]. From these results, the function $e_p^*(\lambda,S)$ can be interpreted as the LCT of the time-reverse early exercise premium $\widetilde{e}_p(\tau,\widetilde{S}_\tau)=e_p(T-\tau,S_{T-\tau})=e_p(t,S_t)$ for $S_t\equiv S$.

In much the same way, we can derive the LCT $C^*(\lambda,S)$ for the American call value: Solving the ODE

$$\frac{1}{2}\sigma^2S^2\frac{\mathrm{d}^2C^*}{\mathrm{d}S^2}+(r-\delta)S\frac{\mathrm{d}C^*}{\mathrm{d}S}-(\lambda+r)C^*+\lambda(S-K)^+=0,\quad S<B_c^*, \tag{3.16}$$

together with the boundary conditions

$$\left|\begin{array}{l}\lim\limits_{S\downarrow 0}C^*(\lambda,S)=0\\ \lim\limits_{S\uparrow B_c^*}C^*(\lambda,S)=B_c^*-K\\ \lim\limits_{S\uparrow B_c^*}\dfrac{\mathrm{d}C^*}{\mathrm{d}S}=1,\end{array}\right. \tag{3.17}$$

where $B_c^*\equiv B_c^*(\lambda)=\mathcal{LC}[\widetilde{B}_c(\tau)]$, we have

Theorem 2 *The LCT* $C^*(\lambda,S)$ *for the American call value is given by*

$$C^*(\lambda,S)=\begin{cases}S-K, & S\geq B_c^*\\ c^*(\lambda,S)+e_c^*(\lambda,S), & S<B_c^*,\end{cases}$$

where

$$e_c^*(\lambda,S)=\frac{1}{\theta_1}\left\{\frac{\delta}{\lambda+\delta}B_c^*-\theta_2\eta(B_c^*)\right\}\left(\frac{S}{B_c^*}\right)^{\theta_1},\quad S<B_c^*,$$

and B_c^* $(\leq K)$ *is a unique positive solution of the functional equation*

$$\lambda\left(\frac{B_c^*}{K}\right)^{\theta_2}+\delta\theta_2\frac{B_c^*}{K}+r(1-\theta_2)=0. \tag{3.18}$$

The function $e_c^*(\lambda, S)$ also can be interpreted as the LCT of the time-reverse early exercise call premium $\widetilde{e}_c(\tau, \widetilde{S}_\tau) = e_c(T-\tau, S_{T-\tau}) = e_c(t, S_t)$ for $S_t \equiv S$, which has the integral representation

$$e_c(t, S_t) = \int_t^T \{\delta S_t \mathrm{e}^{-\delta(u-t)} \Phi(d_+(S_t, B_c(u), u-t)) \\ -rK\mathrm{e}^{-r(u-t)}\Phi(d_-(S_t, B_c(u), u-t))\}\,\mathrm{d}u; \tag{3.19}$$

see Kwok [21, p. 277]

4 Asymptotic Approximations

4.1 Asymptotic Properties

The initial-value theorem in the theory of Laplace transforms

$$\lim_{\lambda\to\infty} B_p^*(\lambda) = \lim_{\tau\to 0} \widetilde{B}_p(\tau) = B_p(T),$$

leads to

Proposition 3 (Kimura [20]) *For the early exercise boundaries of the American put and call options, we have*

$$B_p(T) = \min\left(\frac{r}{\delta}, 1\right) K \quad \textit{and} \quad B_c(T) = \max\left(\frac{r}{\delta}, 1\right) K.$$

See also Kwok [21, pp. 256–262] for another proof.

From the functional equations (3.12) and (3.18) for the LCTs $B_p^*(\lambda)$ and $B_c^*(\lambda)$ in Theorems 1 and 2, we have

Lemma 1 *For sufficiently small $\lambda > 0$,*

$$B_p^*(\lambda) \sim \frac{r}{\delta}\frac{\theta_1 - 1}{\theta_1} K \quad \textit{or} \quad B_p^*(\lambda) \sim \frac{\theta_2}{\theta_2 - 1} K,$$
$$B_c^*(\lambda) \sim \frac{r}{\delta}\frac{\theta_2 - 1}{\theta_2} K \quad \textit{or} \quad B_c^*(\lambda) \sim \frac{\theta_1}{\theta_1 - 1} K.$$

Proof. From (3.12) and (3.18), we immediately obtain

$$B_p^*(\lambda) \sim \frac{r}{\delta}\frac{\theta_1 - 1}{\theta_1} K \quad \text{and} \quad B_c^*(\lambda) \sim \frac{r}{\delta}\frac{\theta_2 - 1}{\theta_2} K,$$

by removing the first terms of the functional equations (3.12) and (3.18). Applying the basic relations into (3.12)

$$\begin{cases} \lambda + r = -\frac{1}{2}\sigma^2\theta_1\theta_2 \\ r - \delta = -\frac{1}{2}\sigma^2(\theta_1 + \theta_2 - 1), \end{cases} \tag{4.1}$$

we have another expression of the equation (3.12) for B_p^*, which is

$$\lambda\left(1-\frac{r-\delta}{\lambda+r}\theta_2\right)\left(\frac{B_p^*}{K}\right)^{\theta_1}+\delta(1-\theta_2)\frac{B_p^*}{K}+r\theta_2\frac{\lambda+\delta}{\lambda+r}=0. \tag{4.2}$$

Similarly, from (3.18) for B_c^*, we have

$$\lambda\left(1-\frac{r-\delta}{\lambda+r}\theta_1\right)\left(\frac{B_c^*}{K}\right)^{\theta_2}+\delta(1-\theta_1)\frac{B_c^*}{K}+r\theta_1\frac{\lambda+\delta}{\lambda+r}=0. \tag{4.3}$$

Deleting the first terms in (4.2) and (4.3) and using the approximation $(\lambda+\delta)/(\lambda+r)\approx\delta/r$ for sufficiently small λ, we obtain the alternative approximations

$$B_p^*(\lambda)\sim\frac{\theta_2}{\theta_2-1}K \quad\text{and}\quad B_c^*(\lambda)\sim\frac{\theta_1}{\theta_1-1}K.$$

□

Proposition 4 (Kimura [20]) *For the time-reverse early exercise boundaries of the American put and call options, we have*

$$\underline{B}_p\equiv\lim_{\tau\to\infty}\widetilde{B}_p(\tau)=\frac{r}{\delta}\frac{\theta_1^\circ-1}{\theta_1^\circ}K=\frac{\theta_2^\circ}{\theta_2^\circ-1}K,$$
$$\overline{B}_c\equiv\lim_{\tau\to\infty}\widetilde{B}_c(\tau)=\frac{r}{\delta}\frac{\theta_2^\circ-1}{\theta_2^\circ}K=\frac{\theta_1^\circ}{\theta_1^\circ-1}K,$$

where $\theta_i^\circ=\lim_{\lambda\to0}\theta_i(\lambda)$, i.e.,

$$\theta_i^\circ=\frac{1}{\sigma^2}\left\{-(r-\delta-\tfrac{1}{2}\sigma^2)-(-1)^i\sqrt{(r-\delta-\tfrac{1}{2}\sigma^2)^2+2\sigma^2r}\right\},\quad i=1,2.$$

4.2 Put-Call Symmetry

Let $\nu_1\equiv\nu_1(\lambda)>1$ and $\nu_2\equiv\nu_2(\lambda)<0$ be two real roots of the quadratic equation

$$\frac{1}{2}\sigma^2\nu^2+(\delta-r-\tfrac{1}{2}\sigma^2)\nu-(\lambda+\delta)=0, \tag{4.4}$$

i.e.,

$$\nu_i=\frac{1}{\sigma^2}\left\{-(\delta-r-\tfrac{1}{2}\sigma^2)-(-1)^i\sqrt{(\delta-r-\tfrac{1}{2}\sigma^2)^2+2\sigma^2(\lambda+\delta)}\right\},\quad i=1,2.$$

As in the case of $\theta_i(\lambda)$ $(i=1,2)$, denote $\nu_i^\circ=\lim_{\lambda\to0}\nu_i(\lambda)$. Clearly, $\nu_i(\lambda)\equiv\nu_i(\lambda;r,\delta)$ and $\theta_i(\lambda)\equiv\theta_i(\lambda;r,\delta)$ $(i=1,2)$ are symmetric with respect to r and δ, namely, $\theta_i(\lambda;\delta,r)=\nu_i(\lambda;r,\delta)$. For these quantities, we have

Lemma 2 *For $\lambda\geq0$,*

$$\theta_1(\lambda)+\nu_2(\lambda)=1,$$
$$\theta_2(\lambda)+\nu_1(\lambda)=1$$

Proof. We only prove the first equation $\theta_1 + \nu_2 = 1$. The second one follows similarly.

$$\begin{aligned}
\nu_2 &= \frac{1}{\sigma^2}\left\{-(\delta - r - \tfrac{1}{2}\sigma^2) - \sqrt{(\delta - r - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+\delta)}\right\} \\
&= 1 - \frac{1}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) + \sqrt{(\delta - r - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+r) + 2\sigma^2(\delta - r)}\right\} \\
&= 1 - \frac{1}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) + \sqrt{(r - \delta - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+r)}\right\} \\
&= 1 - \theta_1,
\end{aligned}$$

and hence $\theta_1(\lambda) + \nu_2(\lambda) = 1$ for $\lambda \geq 0$. □

Proposition 5 *Between two LCTs $B_p^*(\lambda) \equiv B_p^*(\lambda; r, \delta)$ and $B_c^*(\lambda) \equiv B_c^*(\lambda; r, \delta)$ for sufficiently small $\lambda > 0$, there exists a symmetric relation, i.e.,*

$$B_c^*(\lambda; r, \delta) B_p^*(\lambda; \delta, r) \sim K^2.$$

In particular, letting $\lambda \to 0+$, we have

$$\overline{B}_c(r, \delta) \underline{B}_p(\delta, r) = K^2.$$

4.3 Exponential Approximations

Lemma 3 *For sufficiently small $\lambda > 0$,*

$$\begin{aligned}
\theta_1(\lambda) &= \theta_1^\circ + \frac{2}{\sigma^2}\frac{\lambda}{\theta_1^\circ - \theta_2^\circ} + o(\lambda), \\
\theta_2(\lambda) &= \theta_2^\circ + \frac{2}{\sigma^2}\frac{\lambda}{\theta_2^\circ - \theta_1^\circ} + o(\lambda).
\end{aligned}$$

Proof. For simplicity, denote $\alpha \equiv r - \delta - \frac{1}{2}\sigma^2$. Then, for $i = 1, 2$ and sufficiently small $\lambda > 0$, we have

$$\begin{aligned}
\theta_i(\lambda) &= \frac{1}{\sigma^2}\left\{-\alpha - (-1)^i\sqrt{\alpha^2 + 2\sigma^2(\lambda + r)}\right\} \\
&= \frac{1}{\sigma^2}\left\{-\alpha - (-1)^i\sqrt{\alpha^2 + 2\sigma^2 r}\sqrt{1 + \frac{2\sigma^2\lambda}{\alpha^2 + 2\sigma^2 r}}\right\} \\
&= \frac{1}{\sigma^2}\left\{-\alpha - (-1)^i\sqrt{\alpha^2 + 2\sigma^2 r}\left(1 + \frac{\sigma^2\lambda}{\alpha^2 + 2\sigma^2 r}\right)\right\} + o(\lambda) \\
&= \theta_i^\circ - (-1)^i\frac{\lambda}{\sqrt{\alpha^2 + 2\sigma^2 r}} + o(\lambda) \\
&= \theta_i^\circ - (-1)^i\frac{2}{\sigma^2}\frac{\lambda}{\theta_1^\circ - \theta_2^\circ} + o(\lambda),
\end{aligned}$$

where we have used the relation $\theta_1^\circ - \theta_2^\circ = \frac{2}{\sigma^2}\sqrt{\alpha^2 + 2\sigma^2 r}$. □

From Lemmas 1 and 3 and the consistency with the exact boundary values at maturity shown in Proposition 3, we propose the following exponential approximations:

Theorem 3 *For sufficiently large $\tau > 0$, the time-reverse early exercise boundaries have the asymptotically exponential expressions as follows:*

(i) For the American put option,

$$\frac{\widetilde{B}_p(\tau)}{\underline{B}_p} \approx \beta_p(\tau) \equiv \begin{cases} 1 + \dfrac{1}{\theta_1^\circ - 1}\exp\Big\{-\frac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)\tau\Big\}, & r < \delta \\ 1 - \dfrac{1}{\theta_2^\circ}\exp\Big\{-\frac{1}{2}\sigma^2(1-\theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)\tau\Big\}, & r \geq \delta. \end{cases}$$

(ii) For the American call option,

$$\frac{\widetilde{B}_c(\tau)}{\overline{B}_c} \approx \beta_c(\tau) \equiv \begin{cases} 1 + \dfrac{1}{\theta_2^\circ - 1}\exp\Big\{-\frac{1}{2}\sigma^2\theta_2^\circ(\theta_2^\circ - \theta_1^\circ)\tau\Big\}, & r > \delta \\ 1 - \dfrac{1}{\theta_1^\circ}\exp\Big\{-\frac{1}{2}\sigma^2(1-\theta_1^\circ)(\theta_2^\circ - \theta_1^\circ)\tau\Big\}, & r \leq \delta. \end{cases}$$

5 A Heuristic Refinement

Evans, Kuske and Keller [12] have derived explicit expressions valid near expiry for the EEBs of American put and call options, which are, as $\tau \to 0+$,

$$\frac{\widetilde{B}_p(\tau)}{\underline{B}_p} \sim \begin{cases} 1 - \sigma\sqrt{\tau\ln\left(\dfrac{\sigma^2}{8\pi(r-\delta)^2\tau}\right)}, & r > \delta \\ 1 - \sigma\sqrt{2\tau\ln\left(\dfrac{1}{4\sqrt{\pi}\delta\tau}\right)}, & r = \delta \\ 1 - \kappa\sigma\sqrt{2\tau}, & r < \delta, \end{cases} \tag{5.1}$$

and

$$\frac{\widetilde{B}_c(\tau)}{\overline{B}_c} \sim \begin{cases} 1 + \sigma\sqrt{\tau\ln\left(\dfrac{\sigma^2}{8\pi(r-\delta)^2\tau}\right)}, & r < \delta \\ 1 + \sigma\sqrt{2\tau\ln\left(\dfrac{1}{4\sqrt{\pi}\delta\tau}\right)}, & r = \delta \\ 1 + \kappa\sigma\sqrt{2\tau}, & r > \delta, \end{cases} \tag{5.2}$$

where the constant $\kappa \approx 0.4517$ is the root of the transcendental equation

$$\int_\kappa^\infty \mathrm{e}^{-(x^2-\kappa^2)}\mathrm{d}x = \frac{2\kappa^2 - 1}{4\kappa^3}.$$

Clearly, the exponential approximations in Theorem 3 display different tangent behavior near expiry, e.g., for $r < \delta$,

$$\lim_{\tau\to 0+}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\widetilde{B}_p(\tau)}{\underline{B}_p}\right) \approx \beta_p'(0) = -\frac{\sigma^2}{2}\frac{\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)}{\theta_1^\circ - 1} < 0,$$

whereas the exact value is $-\infty$; see Figure 1. This may implies that our approximations for put (call) tend to overestimate (underestimate) the true values for small $\tau > 0$. The asymptotic properties near expiry seems to be helpful for refining our approximations. However, the exact asymptotic expressions above cannot be directly applied to generating refined approximations for EEBs, because if $r \geq \delta$ ($r \leq \delta$) for the put (call) case, (i) they cannot be defined for all $\tau > 0$; and (ii) for the region of τ where they can be defined, they are not monotone functions of τ, being inconsistent with the exact results.

In order to eliminate this defect, we use a simple but rough approximation presented earlier by Barone-Adesi and Whaley [3, Equations (33) and (A10)]. The idea of their approximation was also based on an asymptotic behavior near expiry. With a minor modification of Bjerksund and Stensland [5] on the boundary value at maturity, it is given by, for put

$$\widetilde{B}_p(\tau) \approx \mathrm{e}^{-h_p(\tau)} B_p(T) + \left(1 - \mathrm{e}^{-h_p(\tau)}\right) \underline{B}_p \tag{5.3}$$

where

$$h_p(\tau) = \frac{B_p(T)}{B_p(T) - \underline{B}_p} \left\{-(r-\delta)\tau + 2\sigma\sqrt{\tau}\right\},$$

and for call

$$\widetilde{B}_c(\tau) \approx \mathrm{e}^{-h_c(\tau)} B_c(T) + \left(1 - \mathrm{e}^{-h_c(\tau)}\right) \overline{B}_c \tag{5.4}$$

where

$$h_c(\tau) = \frac{B_c(T)}{\overline{B}_c - B_c(T)} \left\{(r-\delta)\tau + 2\sigma\sqrt{\tau}\right\}.$$

As shown in Barone-Adesi and Whaley [3, p. 310], their approximations also have the same defect on the monotonicity as in (5.1) and (5.2), depending on the values of r and δ. It is, however, relatively easy to eliminate this defect from (5.3) and (5.4). For sufficiently small $\tau > 0$, we have

$$h_p(\tau) \approx \frac{2B_p(T)}{B_p(T) - \underline{B}_p}\sigma\sqrt{\tau} \quad \text{and} \quad h_c(\tau) \approx \frac{2B_c(T)}{\overline{B}_c - B_c(T)}\sigma\sqrt{\tau},$$

which are positive for all $\tau > 0$, and hence they keep the monotonous properties of EEBs. These approximations and the results in Propositions 3 and 4 yields refined approximations of the time-reverse early exercise boundaries for the American put and call options, which are

$$\widetilde{B}_p(\tau) \approx \mathrm{e}^{-\gamma_p(\tau)} B_p(T) + \left(1 - \mathrm{e}^{-\gamma_p(\tau)}\right) \underline{B}_p \beta_p(\tau), \tag{5.5}$$

for put, where

$$\gamma_p(\tau) = \begin{cases} 2\theta_1^\circ \sigma\sqrt{\tau}, & r < \delta \\ 2(1-\theta_2^\circ)\sigma\sqrt{\tau}, & r \geq \delta, \end{cases}$$

and for call

$$\widetilde{B}_c(\tau) \approx \mathrm{e}^{-\gamma_c(\tau)} B_c(T) + \left(1 - \mathrm{e}^{-\gamma_c(\tau)}\right) \overline{B}_c \beta_c(\tau), \tag{5.6}$$

where

$$\gamma_c(\tau) = \begin{cases} -2\theta_2^\circ \sigma\sqrt{\tau}, & r > \delta \\ -2(1-\theta_1^\circ)\sigma\sqrt{\tau}, & r \leq \delta. \end{cases}$$

Note that both exponents $\gamma_p(\tau)$ and $\gamma_c(\tau)$ are nonnegative and increasing functions of $\tau \geq 0$.

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